First-Order Logics and Truth Degrees

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The Inebriation Principle

"There is someone at the party such that if he or she is drunk, then everyone is drunk."

$$(\exists x)(D(x) \rightarrow (\forall y)D(y))$$

Either everyone is drunk and we choose anyone, or someone is not drunk and – *classically* – if this person is drunk, then everyone is drunk.

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Classical Inebriation

By **prenexation**,

$$(\exists x)(D(x) \to (\forall y)D(y)) \quad \text{is equivalent to} \quad (\exists x)(\forall y)(D(x) \to D(y)),$$

which, by **Skolemization**, is valid if and only if

$$(\exists x)(D(x) \to D(f(x)))$$

is valid, which, using **Herbrand's theorem**, it is, because

$$(D(c) \to D(f(c))) \ \lor \ (D(f(c)) \to D(f(f(c))))$$

is propositionally valid in classical logic.

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Many-Valued Inebriation

How does our inebriation principle fare with respect to. . .

. . . truth values in $[0,1]$

. . . truth $\top$ as 1 and falsity $\perp$ as 0

. . . $\wedge$ as minimum and $\vee$ as maximum

. . . $\forall$ as inf and $\exists$ as sup ?

Roughly, the principle holds for *finitely* many truth values, but for *infinitely* many values, it depends on how we interpret **implication**.

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Two Logics

Gödel vs Łukasiewicz

Ordered Inebriation

Given truth values $[0, 1]$ and the **Gödel implication**

$$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise,} \end{cases}$$

the inebriation principle $(\exists x)(D(x) \to (\forall y)D(y))$ *fails*.

Suppose that there are people $p_1, p_2, \ldots$ and that each person p_n is drunk to degree $\frac{1}{n}$. Then "everyone is drunk" is completely false, as is "p_n is drunk implies everyone is drunk". So the inebriation principle has degree 0.

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Continuous Inebriation

Given truth values $[0, 1]$ and the **Łukasiewicz implication**

$$a \to b = \min(1, 1 - a + b),$$

the inebriation principle $(\exists x)(D(x) \to (\forall y)D(y))$ *holds*.

Suppose that everybody is drunk to at least degree k and that k is the greatest truth value with this property. Then for each $\epsilon > 0$, there is a person p_ϵ drunk to at most degree $k + \epsilon$ and "p_ϵ is drunk implies everyone is drunk" has degree at least $1 - \epsilon$. So the inebriation principle has value 1.

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Logical consequence in first-order classical logic,

$$\Gamma \models \varphi,$$

reduces to the **unsatisfiability** of

- $\Gamma \cup \{\neg\varphi\}$ via **double-negation** and the **deduction theorem**
- a set of prenex formulas via **quantifier shifts**
- a set of universal formulas via **Skolemization**
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Main Question

Which of these steps can be applied in the many-valued setting?

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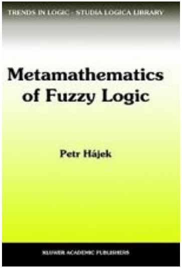
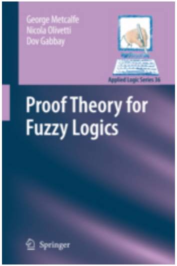
Disclaimer

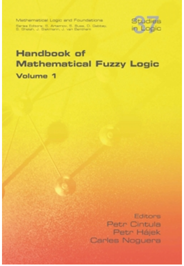
This talk is based partly on

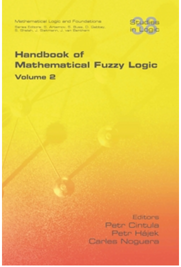
> P. Cintula and G. Metcalfe. Herbrand Theorems for Substructural Logics. *Proceedings of LPAR 2013*, LNCS 8312, Springer, 584–600.

For further details and references see









Beyond Two Values

We consider a propositional language $\mathcal{L}$ and (classes of) $\mathcal{L}$-algebras

$$\mathbf{A} = \langle A, \wedge, \vee, \{\star_i\}_{i \in I}, \bot, \top \rangle$$

where $\langle A, \wedge, \vee, \bot, \top \rangle$ is a **complete chain** (e.g., $\{0, 1\}$, $[0, 1]$, $\mathbb{N} \cup \{\infty\}$).

Order-Based Algebras

For Gödel implication on **A**, only the **order** of values matters...

$$a \to b \;=\; \begin{cases} \top & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

and similarly for operations such as

$$\Delta a \;=\; \begin{cases} \top & \text{if } a = \top \\ \bot & \text{otherwise} \end{cases} \qquad \text{and} \qquad a \leftarrow b \;=\; \begin{cases} \bot & \text{if } b \leq a \\ b & \text{otherwise.} \end{cases}$$

More formally, such operations are definable in **A** by a quantifier-free formula in the first-order language with only $\wedge$, $\vee$, and constants of $\mathcal{L}$.

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The Standard Łukasiewicz Algebra

Łukasiewicz implication on $[0,1]$ is the **continuous** function

$$a \to b = \min(1, 1 - a + b).$$

The functions min and max are definable using $\to$ and 0, as are

$$\neg a = 1 - a \qquad \text{and} \qquad a \oplus b = \min(1, a + b).$$

Indeed, interpretations of formulas relate 1-1 to piecewise linear continuous functions on $[0,1]$ with integer coefficients (McNaughton 1951).

Other "logics of continuous functions" include **rational Pavelka logic**, **continuous logic**, and **abelian logic**.

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First-Order Languages

A **predicate language** $\mathcal{P}$ is a triple $\langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$ where $\mathbf{P}$ and $\mathbf{F}$ are non-empty sets of predicate symbols and function symbols, respectively, and $\mathbf{ar}$ is a function assigning to $\star \in \mathbf{P} \cup \mathbf{F}$ an *arity* $\mathbf{ar}(\star) \in \mathbb{N}$.

$\mathcal{P}$-**terms** $s, t, \ldots$ and $\mathcal{P}$-**formulas** $\varphi, \psi, \ldots$ are defined as in classical logic using a fixed countably infinite set OV of **object variables** $x, y, \ldots$, propositional connectives from $\mathcal{L}$, and the quantifiers $\forall$ and $\exists$.

A $\mathcal{P}$-**theory** Γ is just a set of $\mathcal{P}$-formulas.

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First-Order Structures

A $\mathcal{P}$-**structure** $\mathfrak{M} = \langle \mathbf{A}, \mathbf{M} \rangle$ consists of an $\mathcal{L}$-algebra $\mathbf{A}$ based on a complete chain, and

$$\mathbf{M} = \langle M, \{P^{\mathbf{M}}\}_{P \in \mathbf{P}}, \{f^{\mathbf{M}}\}_{f \in \mathbf{F}} \rangle$$

where M is a non-empty set, $P^{\mathbf{M}} \colon M^n \to A$ is a function for each n-ary $P \in \mathbf{P}$, and $f^{\mathbf{M}} \colon M^n \to M$ is a function for each n-ary $f \in \mathbf{F}$.

Evaluations

Given an $\mathfrak{M}$-**evaluation** v mapping object variables to M,

$$\begin{array}{rcll}
\|x\|^{\mathfrak{M}}_{\mathrm{v}} & = & \mathrm{v}(x) & (x \in OV) \\
\|f(t_1,\ldots,t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & f^{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}},\ldots,\|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) & (f \in \mathbf{F}) \\
\|P(t_1,\ldots,t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & P^{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}},\ldots,\|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) & (P \in \mathbf{P}) \\
\|\star(\varphi_1,\ldots,\varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & \star^{\mathbf{A}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}},\ldots,\|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}) & (\star \in \mathcal{L}),
\end{array}$$

and letting $\mathrm{v}[x{\to}a](x) = a$ and $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ for $y \neq x$,

$$\begin{array}{rcl}
\|(\forall x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} & = & \bigwedge^{\mathbf{A}}_{a\in M} \|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to a]} \\
\|(\exists x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} & = & \bigvee^{\mathbf{A}}_{a\in M} \|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to a]}.
\end{array}$$

Notation. Given $\mathrm{v}(\vec{x}) = \vec{a}$, we often write $\|\varphi(\vec{a})\|^{\mathfrak{M}}$ for $\|\varphi(\vec{x})\|^{\mathfrak{M}}_{\mathrm{v}}$.

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$$\begin{array}{rcl} \|(\forall x)\varphi\|_{\mathrm{v}}^{\mathfrak{M}} & = & \bigwedge^{\mathbf{A}}_{a \in M} \|\varphi\|_{\mathrm{v}[x\to a]}^{\mathfrak{M}} \\ \|(\exists x)\varphi\|_{\mathrm{v}}^{\mathfrak{M}} & = & \bigvee^{\mathbf{A}}_{a \in M} \|\varphi\|_{\mathrm{v}[x\to a]}^{\mathfrak{M}}. \end{array}$$

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$$\begin{array}{rcll} \|x\|^{\mathfrak{M}}_{\mathrm{v}} & = & \mathrm{v}(x) & (x \in OV) \\ \|f(t_1,\ldots,t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & f^{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}},\ldots,\|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) & (f \in \mathbf{F}) \\ \|P(t_1,\ldots,t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & P^{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}},\ldots,\|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) & (P \in \mathbf{P}) \\ \|\star(\varphi_1,\ldots,\varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & \star^{\mathbf{A}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}},\ldots,\|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}) & (\star \in \mathcal{L}), \end{array}$$

and letting $\mathrm{v}[x{\to}a](x) = a$ and $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ for $y \neq x$,

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Notation. Given $\mathrm{v}(\vec{x}) = \vec{a}$, we often write $\|\varphi(\vec{a})\|^{\mathfrak{M}}$ for $\|\varphi(\vec{x})\|^{\mathfrak{M}}_{\mathrm{v}}$.

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Models

Let $\mathbb{K}$ be a class of $\mathcal{L}$-algebras based on complete chains and let $\mathbf{A} \in \mathbb{K}$. Then a $\mathcal{P}$-structure $\mathfrak{M} = \langle \mathbf{A}, \mathbf{M} \rangle$ is a $\mathbb{K}$-**model** of a $\mathcal{P}$-theory Γ, written

$$\mathfrak{M} \models \Gamma,$$

if $\|\varphi\|_{\mathrm{v}}^{\mathfrak{M}} = \top^{\mathbf{A}}$ for each $\varphi \in \Gamma$ and $\mathfrak{M}$-evaluation v.

Note. A $\mathcal{P}$-structure $\mathfrak{M} = \langle \mathbf{A}, \mathbf{M} \rangle$ is called **witnessed** if for each $\mathcal{P}$-formula $\varphi(x, \vec{y})$ and $\vec{a} \in M$, there exist $b, c \in M$ such that

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If $\Gamma \cup \{\varphi\}$ is a $\mathcal{P}$-theory and for each $\mathbf{A} \in \mathbb{K}$ and $\mathcal{P}$-structure $\mathfrak{M} = \langle \mathbf{A}, \mathbf{M} \rangle$,

$$\mathfrak{M} \models \Gamma \quad \Longrightarrow \quad \mathfrak{M} \models \varphi,$$

then we say that "φ is a **consequence** of Γ in $\mathbb{K}$", written

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First-Order Gödel Logic

When $\mathbb{K}$ consists of just the standard Gödel algebra on $[0,1]$, we obtain **first-order Gödel logic** (and write $\models_{\mathrm{G}}$) which

- admits the **deduction theorem**

$$\Gamma \cup \{\varphi\} \models_{\mathrm{G}} \psi \quad \Leftrightarrow \quad \Gamma \models_{\mathrm{G}} \varphi \to \psi \qquad (\varphi, \psi \text{ sentences})$$

  but not double negation, i.e., $\not\models_{\mathrm{G}} \neg\neg\varphi \to \varphi$

- admits some **quantifier shifts** such as

$$\models_{\mathrm{G}} ((\exists x)\varphi \to \psi) \leftrightarrow (\forall x)(\varphi \to \psi) \quad (x \text{ not free in } \psi)$$

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$$\text{(prl)} \quad (\varphi \to \psi) \lor (\psi \to \varphi)$$

$$\text{(cd)} \quad (\forall x)(\varphi \lor \psi) \to (\varphi \lor (\forall x)\psi) \quad (x \text{ not free in } \varphi)$$

- is **finitary**, i.e.,

$$\Gamma \models_{\mathrm{G}} \varphi \qquad \Leftrightarrow \qquad \Gamma' \models_{\mathrm{G}} \varphi \ \text{ for some finite } \Gamma' \subseteq \Gamma$$

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First-Order Gödel *Logics*

Remark. Restricting the standard Gödel algebra to a closed infinite subset of $[0, 1]$ containing $\{0, 1\}$, we obtain countably infinitely many different first-order Gödel logics (Beckmann, Goldstern, and Preining 2008).

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When $\mathbb{K}$ consists of the standard Łukasiewicz algebra on $[0, 1]$, we obtain **first-order Łukasiewicz logic** (and write $\models_{\text{Ł}}$), which

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- is not **(recursively) axiomatizable** (Scarpellini 1962), and indeed is Π_2-complete (Ragaz 1981)
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Other Possibilities...

- If $\mathbb{K}$ consists of the standard product algebra on $[0, 1]$ with

$$a \cdot b = ab \qquad \text{and} \qquad a \to b = \begin{cases} 1 & a \leq b \\ \frac{b}{a} & \text{otherwise,} \end{cases}$$

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- If $\mathbb{K}$ consists of the complete chains of certain varieties of **integral commutative residuated lattices**, then we obtain a (recursively) axiomatizable and finitary first-order logic (e.g., **first-order MTL**).

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Half-Time Score

	Classical	Gödel	Łukasiewicz	Product
Double Negation	YES	NO	YES	NO
Deduction Theorem	YES	YES	NO	NO
Prenex Forms	YES	NO	YES	NO
Witnessed Models	YES	NO	YES	NO
Axiomatizable	YES	YES	NO	NO
Finitary	YES	YES	NO	NO

Skolemization Right

Theorem

Let $\mathbb{K}$ be a class of $\mathcal{L}$-algebras based on complete chains. For any $\mathcal{P}$-theory $\Gamma \cup \{\varphi(x, \vec{y})\}$ and function symbol $f_\varphi \notin \mathcal{P}$ with arity $|\vec{y}|$:

$$\Gamma \models_{\mathbb{K}} (\exists \vec{y})(\forall x)\varphi(x, \vec{y}) \quad \Leftrightarrow \quad \Gamma \models_{\mathbb{K}} (\exists \vec{y})\varphi(f_\varphi(\vec{y}), \vec{y}).$$

Proof.

($\Leftarrow$) Easy, because $(\exists \vec{y})(\forall x)\varphi(x, \vec{y}) \models_{\mathbb{K}} (\exists \vec{y})\varphi(f_\varphi(\vec{y}), \vec{y})$.

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Skolemization Right

Theorem

Let $\mathbb{K}$ be a class of $\mathcal{L}$-algebras based on complete chains. For any $\mathcal{P}$-theory $\Gamma \cup \{\varphi(x, \vec{y})\}$ and function symbol $f_\varphi \notin \mathcal{P}$ with arity $|\vec{y}|$:

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Skolemization Left

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Let $\mathbb{K}$ consist of the standard Łukasiewicz or Gödel algebra. For any $\mathcal{P}$-theory $\Gamma \cup \{\varphi(x, \vec{y}), \psi\}$ and function symbol $f_\varphi \notin \mathcal{P}$ with arity $|\vec{y}|$:

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Moreover, first-order Łukasiewicz logic has prenexation, so in this case we can Skolemize all formulas on both sides of the consequence relation.

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Skolemization Left and Regular Completions

A class $\mathbb{K}$ of integral commutative residuated lattices admits **regular completions** if for each $\mathbf{A} \in \mathbb{K}$, there is an embedding of $\mathbf{A}$ into some complete $\mathbf{B} \in \mathbb{K}$ that preserves infinite meets and joins when they exist.

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Let $\mathbb{K}$ be the class of complete chains of a variety of integral commutative residuated lattices whose class of chains admits regular completions. For any $\mathcal{P}$-theory $\Gamma \cup \{\varphi(x, \vec{y}), \psi\}$ and function symbol $f_\varphi \notin \mathcal{P}$ with arity $|\vec{y}|$:

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A Herbrand Theorem for First-Order Gödel Logic

The **Herbrand universe** $\mathcal{U}(\mathcal{P})$ for a predicate language $\mathcal{P}$ with at least one constant is the set of ground (variable-free) $\mathcal{P}$-terms.

Theorem

For any quantifier-free $\mathcal{P}$-formula $\varphi(\vec{x})$:

$$\models_{\mathrm{G}} (\exists \vec{x})\varphi(\vec{x}) \quad \Leftrightarrow \quad \models_{\mathrm{G}} \bigvee_{i=1}^{n} \varphi(\vec{t}_i) \text{ for some } \vec{t}_1, \ldots, \vec{t}_n \in \mathcal{U}(\mathcal{P}).$$

This has been proved proof-theoretically by Baaz and Zach (2000) and semantically by Baaz, Ciabattoni, and Fermüller (2001).

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An Expansion Lemma

Let Δ_0 denote the quantifier-free $\mathcal{P}$-formulas, and define using BNF:

$$\begin{array}{rrcl} \textbf{g-universal formulas} & P ::= & \Delta_0 & \mid P \wedge P \mid P \vee P \mid (\forall x)P \mid N \to P \\ \textbf{g-existential formulas} & N ::= & \Delta_0 & \mid N \wedge N \mid N \vee N \mid (\exists x)N \mid P \to N. \end{array}$$

We refer to theories containing only g-universal and g-existential formulas as **g-universal** and **g-existential theories**, respectively.

Lemma

For each g-existential $\mathcal{P}$-formula ψ and g-universal $\mathcal{P}$-theory $\Gamma \cup \{\varphi\}$:

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Herbrand Expansions

The $\mathcal{P}$-**Herbrand expansion** $E(\varphi)$ of a $\mathcal{P}$-formula φ consists of formulas obtained by applying the following steps repeatedly:

(1) Replace $\varphi[(\forall\vec{x})\psi(\vec{x},\vec{y})]$ where ψ is quantifier-free with $\varphi[\bigwedge_{\vec{t}\in H}\psi(\vec{t},\vec{y})]$ for some finite $H\subseteq\mathcal{U}(\mathcal{P})$.

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More General Herbrand Theorems

Theorem

For every g-universal $\mathcal{P}$-theory $\Gamma \cup \{\varphi\}$ and g-existential $\mathcal{P}$-formula ψ:

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These theorems hold for any **finitary** $\mathbb{K}$ – e.g., if $\mathbb{K}$ is the class of complete chains of a variety of integral commutative residuated lattices whose class of chains admits regular completions (see also recent work of Terui).

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Failure of the Herbrand Theorem in Łukasiewicz Logic

Easily, by **prenexation**,

$$\models_{Ł} (\exists x)(\forall y)(D(x) \to D(y)).$$

So, by **Skolemization right**,

$$\models_{Ł} (\exists x)(D(f(x)) \to D(x)).$$

For **Herbrand's Theorem**, we would need for some $n \in \mathbb{N}$,

$$\models_{Ł} \bigvee_{i=1}^{n} (D(f^{i+1}(c)) \to D(f^{i}(c))),$$

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Approximate Consequence

Let us define for a $\mathcal{P}$-theory Γ, $\mathcal{P}$-sentence ψ, and $r \in [0,1] \cap \mathbb{Q}$,

$$\Gamma \models_{Ł} r < \psi \qquad \Leftrightarrow \qquad r < \|\psi\|^{\mathfrak{M}} \text{ for every model } \mathfrak{M} \text{ of } \Gamma,$$

and note that if the "propositional atom" P does not occur in $\Gamma \cup \{\psi\}$,

$$\Gamma \models_{Ł} \tfrac{n}{n+1} < \psi \qquad \Leftrightarrow \qquad \Gamma \cup \{\psi \to (P \wedge \neg P^n)\} \models_{Ł} \bot.$$

First-order Łukasiewicz logic is not finitary, but

$$\Gamma \models_{Ł} \bot \qquad \Leftrightarrow \qquad \Gamma' \models_{Ł} \bot \text{ for some finite } \Gamma' \subseteq \Gamma.$$

Moreover, for every universal theory $\Gamma \cup \{\varphi\}$,

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Approximate Herbrand Theorems

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Theorem

For each universal theory $\Gamma \cup \{\varphi\}$ *and existential formula* ψ:

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Approximate Herbrand Theorems

Theorem

For any universal theory $\Gamma \cup \{\varphi\}$, *existential formula* ψ, *and* $r \in [0,1] \cap \mathbb{Q}$:

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So Which Logic Wins?

First-order Gödel logic is axiomatizable, has the deduction theorem, and admits Skolemization and Herbrand theorems for prenex formulas. . .

First-order Łukasiewicz logic is not axiomatizable but admits prenex forms and therefore Skolemization for all formulas and a reduction – via an approximate Herbrand theorem – to propositional logic.

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Using Skolemization and the approximate Herbrand theorem, for any quantifier-free and function-symbol-free formula $\varphi(\vec{x}, \vec{y})$:

$$\begin{aligned}\models_{Ł} (\forall \vec{x})(\exists \vec{y})\varphi(\vec{x}, \vec{y}) \quad &\Leftrightarrow \quad \models_{Ł} (\exists \vec{y})\varphi(\vec{c}, \vec{y}) \ \text{ for new constants } \vec{c} \\ &\Leftrightarrow \quad \models_{Ł} r < \bigvee_{\vec{d} \subseteq \vec{c}} \varphi(\vec{c}, \vec{d}) \ \text{ for each } r \in [0,1) \\ &\Leftrightarrow \quad \models_{Ł} \bigvee_{\vec{d} \subseteq \vec{c}} \varphi(\vec{c}, \vec{d}).\end{aligned}$$

Corollary (Rutledge 1959)

Validity in the one-variable fragment of first-order Łukasiewicz logic is decidable (indeed co-NP complete).

Validity in the **monadic fragment** is undecidable (Bou), but validity in certain **modal (description logic) fragments** is decidable (Hájek 2005).

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The **monadic fragment** of first-order Gödel logic is undecidable (Baaz, Ciabattoni, and Fermüller 2007).

Validity in the **one-variable fragment** does not have the finite model property; e.g.,

$$(\forall x)(\neg\neg P(x)) \to \neg\neg(\forall x)P(x)$$

is valid in all finite models, but not in the model with universe $\mathbb{N}$ where $P(a)$ is interpreted as $\frac{1}{a+1}$ for $a \in \mathbb{N}$.

However, decidability for this and certain **modal fragments** has been established via a new semantics for Gödel modal logics in

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New Quantifiers

Consider the following functions (definable in Gödel logic with Δ):

$$a * b = \begin{cases} 0 & a > b \\ 1 & b = 1 \\ a & \text{otherwise} \end{cases} \qquad \text{and} \qquad a \circ b = \begin{cases} 1 & a < b \\ 0 & b = 0 \\ a & \text{otherwise.} \end{cases}$$

For a one-variable formula φ and a predicate symbol P not occurring in φ, let $\bar{\varphi}$ be φ with subformulas $(\forall x)\psi$ and $(\exists x)\psi$ replaced with

$$(\bar{\forall} x)\psi = (\exists y)(P(y) * (\forall x)\psi) \qquad \text{and} \qquad (\bar{\exists} x)\psi = (\forall y)(P(y) \circ (\exists x)\psi).$$

Validity of these formulas $\bar{\varphi}$ has the finite model property, and

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